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Triply periodical particulate matrix composites in varying external stress fields

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Abstract

We consider a linear elastic composite medium, which consists of a homogeneous matrix containing aligned ellipsoidal uncoated or coated inclusions arranged in a periodic array and subjected to inhomogeneous boundary conditions. The hypothesis of effective field homogeneity near the inclusions is used. The general integral equation obtained reduces the analysis of infinite number of inclusion problems to the analysis of a finite number of inclusions in some representative volume element (RVE). The integral equation is solved by the Fourier transform method as well as by the iteration method of the Neumann series (first-order approximation). The nonlocal macroscopic constitutive equation relating the unit cell averages of stress and strain is derived in explicit closed forms either of a differential equation of a second-order or of an integral equation. The employed of explicit relations for numerical estimations of tensors describing the local and nonlocal effective elastic properties as well as average stresses in the composites containing simple cubic lattices of rigid inclusions and voids are considered.

1. Introduction

A considerable number of methods are known in the linear theory of random structure matrix composites which yield the effective elastic constants and stress field averages in the component. Appropriate, but by no means exhaustive, references are provided by the reviews of Willis (1982, 1983), Mura (1987), Kreher and Pompe (1989), Buryachenko and Parton (1992), Nemat-Nasser and Hori (1993), Buryachenko (1996). A new method has been proposed recently. This is the multiparticle effective field method (MEFM), put forward and developed by the present author (references may be found in the survey of Buryachenko and Parton, 1992, Buryachenko and Kreher, 1995). For random structure composites MEFM is based on the theory of functions of random variables and Green's functions. Within this method one constructs a hierarchy of stat-

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istical moment equations for conditional averages of the stresses in the inclusions. The hierarchy is then cut by introducing the notion of an effective field. This way the interaction of different inclusions is taken into account. Buryachenko and Parton (1992) demonstrated that the MEFM includes, as particular cases, the well-known methods of mechanics of strongly heterogeneous media such as the effective medium (Kröner, 1961; Hill, 1965) and the mean field (Mori and Tanaka, 1973; Benveniste, 1987) methods.

This paper deals with composite media which consist of a homogeneous matrix containing the triply periodical sets of inclusions of ellipsoidal form. The periodic structure of composite is very attractive because it provides the estimation of interaction effects for an infinite number of inclusions. This interaction greatly influences the composite elastic properties, especially for the strongly heterogeneous materials with a high inclusion concentration. The periodicity of structures gives the possibility of finding an analytical (or numerical) solution of the corresponding periodic boundary-value problem with controlled accuracy.

For periodic structure composites there are different methods for solving the cell problem. The method of Eshelby transformation strain, taking account of the variability of the field of transformation strain within the inclusion, has been proposed by Nemat-Nasser et al. (1982). However, because they used very slowly converging series, their approach is hardly applicable. Using the periodic fundamental solution for an isotropic medium by Hasimoto (1959) numerical values of the effective characteristics of dispersionally reinforced composites with isotropic components were obtained by Sangani and Lu (1987) by multiple expansions. These authors extended a collocation technique employed by Nunan and Keller (1984). Media with arbitrary elastic anisotropy were considered by Kuznetsov (1993). Kushch (1997) analyzed the elastic isotropic medium containing several triply periodic lattices of aligned spheroidal isotropic inclusions with different size, shape and properties.

By virtue of the fact that a periodic structure is a particle case of random structure, Buryachenko and Parton (1992) applied MEFM to the analysis of periodic structures. They used the main hypothesis of many micromechanical methods, according to which each inclusion is located inside a homogeneous so-called effective field. Note that in the framework of the popular simplification, the surrounding inclusions are simulated by the singular sources of polarization strains located at the centers of inclusions (see e.g. Kunin, 1983). In a proposed method by Buryachenko and Parton (1986), the approximation is in the spirit of the Saint-Venant principle so that the effective field is uniformly distributed inside the inclusion. At the same time Kachanov (1987) proposed a similar approach for a particular case of ellipsoidal cavities (microcracks). Buryachenko and Parton (1992) reduced the system of integral equations to a linear algebraic system of equations with respect to effective fields; the number of unknowns is finite in the case of periodic structure. The final solution was obtained for the general case of coated inclusions and any ellipsoidal shape of a representative volume element (RVE). They justified the locality principle for the particular case of simple cubic packing of the spherical inclusions, when the maximum error of the using of the RVE (in excess of the distance between the inclusions in three times) is smaller than 2%. More recently the same problem has also been considered by the use of some additional unnecessary assumptions. So Rodin (1993) proposed a different equivalent approach based on the eigenstrain method; the convergence of integral representations was justified for media with an isotropic matrix containing homogeneous spherical inclusions. Molinari and Mouden (1996) analyzed the analogous problem for the spherical RVE.

The assumption usually used is that all characteristic lengths associated with the spatial variations of the mean field quantities are large compared to all characteristic lengths associated with the spatial variations in the material properties. Then the governing equations for the mean field are identical in form to the familiar equations for a homogeneous solid with material properties replaced by effective properties. The breakdown of this condition mentioned above leads to a nonlocal coupling between statistical averages for random structures (or average over the cell for periodic structures) of stresses and strains which is represented easier by the integral or by the differential operators. The method of Fourier transform has been investigated in nonlocal micromechanics of random structure composites and was used with the slight modifications by Beran and McCoy (1970), Buryachenko and Lipanov (1992), Drugan and Willis (1996), Khoroshun (1996), Buryachenko (1998). As will be shown in the current paper the same approach can be employed for the analysis of triply periodic structures.

The outline of the paper is as follows. In Section 2 present the basic equation and geometrical description of the composite structure, and two kinds of averaging operators. In Section 3 we derive a general integral equation of elasticity of triply periodic structures subjected to inhomogeneous boundary conditions; the equations obtained reduce the analysis of infinite numbers of inclusion problems to the analysis of a finite number of inclusions. In order to simplify the general integral system one assumes the effective field hypothesis. In Section 4 the entire set of equations is transformed from real space to Fourier transform space and an algebraic solution for the effective average strain field formally obtained in the transformed space; the inverse transformation transforms this algebraic solution into a differential operator of the second order in real space. A common integral representation of the solution is derived by the iteration method. Once the average effective field is obtained, the effective local and nonlocal properties of the composite material are calculated via the homogenized relation in Section 5. Finally, in Section 6 we employ the proposed explicit relations for numerical estimations of tensors describing the local and nonlocal effective elastic properties of composites containing simple cubic lattices of rigid inclusions and voids. The local and nonlocal parts of average stresses are estimated by both the Fourier transform method and by the iteration method.

2. Preliminaries

2.1. Basic equations

The paper discusses a certain representative mesodomain w with a characteristic function W containing a set $X = (v_i)$ of inclusions v_i with characteristic functions V_i $(i = 1, 2, \ldots)$. At first no restrictions are imposed on the elastic symmetry of the phases or on the geometry of the inclusions. It is assumed that the inclusions can be grouped into component $v^{(1)}$ with identical mechanical and geometrical properties. The local strain tensor $\boldsymbol{\varepsilon}$ is related to the displacements $\mathbf{u}$ via the linearized strain–displacement equation

$$\boldsymbol{\varepsilon} = \tfrac{1}{2}[\nabla \otimes \mathbf{u} + (\nabla \otimes \mathbf{u})^{\mathrm{T}}]. \tag{2.1}$$

Here $\otimes$ denotes tensor product, and $(\cdot)^{\mathrm{T}}$ denotes matrix transposition. The stress tensor $\boldsymbol{\sigma}$, satisfies the equilibrium equation (no body forces acting):

$$\nabla\boldsymbol{\sigma} = 0. \tag{2.2}$$

Stresses and strains are related to each other via the constitutive equation

$$\boldsymbol{\sigma}(\mathbf{x}) = \mathbf{L}(\mathbf{x})\boldsymbol{\varepsilon}(\mathbf{x}). \tag{2.3}$$

$\mathbf{L}(\mathbf{x})$ are the known phase stiffness fourth-order tensors, and the common notations for scalar products have been employed: $\mathbf{L}\boldsymbol{\varepsilon} = L_{ijkl}\varepsilon_{kl}$, $\boldsymbol{\varepsilon}\mathbf{x} = \varepsilon_{ij}x_j$. The tensor $\mathbf{L}$ of material properties is decomposed as $\mathbf{L} \equiv \mathbf{L}^{(0)} + \mathbf{L}_1(\mathbf{x})$. $\mathbf{L}$ is assumed to be constant in the matrix $v^{(0)} = w \backslash v$ and is an inhomogeneous function inside the inclusions:

$$\mathbf{L}(\mathbf{x}) = \begin{cases} \mathbf{L}^{(0)} & \text{for } \mathbf{x} \in v^{(0)}, \\ \mathbf{L}^{(0)} + \mathbf{L}_1^{(1)}(\mathbf{x}) & \text{for } \mathbf{x} \in v^{(1)}. \end{cases} \tag{2.4}$$

Here and in the following the upper index (k) $(k = 0, 1)$ numbers the components and the lower index i numbers the individual conclusions; $v^{(1)} \equiv \bigcup v_i$, $(i = 1, 2, \ldots)$.

We assume that the phases are perfectly bonded, so that the displacements and the traction components are continuous across the interphase boundaries. We take nonuniform strain boundary conditions for the mesodomain w

$$\mathbf{u}(\mathbf{x}) = \mathbf{u}^0(\mathbf{x}), \quad \boldsymbol{\varepsilon}^0(\mathbf{x}) \equiv \tfrac{1}{2}[\nabla \otimes \mathbf{u}^0(\mathbf{x}) + (\nabla \otimes \mathbf{u}^0(\mathbf{x}))^{\mathrm{T}}], \quad \mathbf{x} \in \partial w, \tag{2.5}$$

where $\boldsymbol{\varepsilon}^0(\mathbf{x})$ is a given nonuniform symmetric tensor, representing the macroscopic strain state in the mesodomain w if the boundary conditions (2.5) is a homogeneous one:

$$\boldsymbol{\varepsilon}^0(\mathbf{x}) \equiv \boldsymbol{\varepsilon}^0 = \text{const.}, \quad \mathbf{x} \in \partial w. \tag{2.6}$$

2.2. Geometrical description of the composite structure

It is assumed that the representative mesodomain w contains a statistically large number of inclusions $v_i \subset v^{(1)}$ $(i = 1, 2, \ldots)$. We now consider a periodic set X of ellipsoidal inclusions with identical shape, orientation and mechanical properties. We consider a composite media with particle centers, periodically distributed at the nodes of same spatial lattice Λ. Suppose $\mathbf{e}_i$ $(i = 1, 2, 3)$ are linearly-independent vectors of the principal period of Λ determine a unit cell Ω of volume $\bar{\Omega} = |\mathbf{e}_1 \cdot (\mathbf{e}_2 \otimes \mathbf{e}_3)|$, so that we can represent any node $\mathbf{m} \in \Lambda$ in the form

$$\mathbf{x}_{\mathbf{m}} = \Sigma m_i \mathbf{e}_i, \tag{2.7}$$

where $\mathbf{m} = (m_1, m_2, m_3)$ are integer-valued coordinates of the node $\mathbf{m}$ in periodic basis $\mathbf{e}_i$ which are equal in modulus to $|\mathbf{e}_i|$. The bar placed above the region represents its measure: $\bar{\Omega} = \operatorname{mes}\Omega$.

Note that the type of the lattice Λ is defined by the law governing the variation in the coefficients m_i $(i = 1, 2, 3)$, and also by the magnitude and orientation of the vectors $\mathbf{e}_i$ (see e.g. Kuznetsov, 1991). If, for example, the basis is orthonormal, and the coefficients $\mathbf{m} = (m_1, m_2, m_3)$ are the integer set Z^3, independent of one another, Λ defines a simple cubic (SC) packing; in the case where the coefficients m_i $(i = 1, 2, 3)$ are either all even or odd, we have a cubic body-centered structure (BCC); a cubic face-centered structure (FCC) is obtained in that case where the coefficients m_i are either all even or two are odd, while the third is even. The method of assigning the lattice Λ is also

possible where several nodes are located within the limits of a cell, and the coefficients m_i are the integer set Z^3, independent of one another.

The composite material is constructed using the basic building block or generic unit cell: $w = \bigcup \Omega_{\mathbf{m}}$, $v_{\mathbf{m}} \subset \Omega_{\mathbf{m}}$. Hereinafter the notations $\mathbf{f}^{\Omega}(\mathbf{x})$ will be used for the average of the function $\mathbf{f}$ over the cell $\mathbf{x} \in \Omega_i$ with the center $\mathbf{x}_i^{\Omega} \in \Omega_i$:

$$\mathbf{f}^{\Omega}(\mathbf{x}) = \mathbf{f}^{\Omega}(\mathbf{x}_i^{\Omega}) \equiv \frac{1}{\bar{\Omega}_i} \int_{\Omega_i} \mathbf{f}(\mathbf{y})\, d\mathbf{y}, \quad \mathbf{x} \in \Omega_i. \tag{2.8}$$

For the periodic structures $1/\bar{\Omega}_i \equiv n = \text{const.}$, where n is a number density of inclusions.

Let $\mathscr{V}_{\mathbf{x}}$ be a 'moving averaging' cell with the center $\mathbf{x}$ obtained by translation of a cell Ω_i, and let for the sake of definiteness $\boldsymbol{\xi}$ be a random vector uniformly distributed on $\mathscr{V}_{\mathbf{x}}$ whose value at $\mathbf{z} \in \mathscr{V}_{\mathbf{x}}$ is $\varphi_{\xi}(\mathbf{z}) = 1/\bar{\mathscr{V}}_{\mathbf{x}}$ and $\varphi_{\xi}(\mathbf{z}) \equiv 0$ otherwise. Then we can define the average of the function $\mathbf{f}$ with respect to translations of the vector $\boldsymbol{\xi}$

$$\langle \mathbf{f} \rangle_{\mathbf{x}}(\mathbf{x}-\mathbf{y}) = \frac{1}{\bar{\mathscr{V}}_{\mathbf{x}}} \int_{\mathscr{V}_{\mathbf{x}}} \mathbf{f}(\mathbf{z}-\mathbf{y})\, d\mathbf{z}, \quad \mathbf{x} \in \Omega_i. \tag{2.9}$$

Let $\mathbf{f}$ be governed by the boundary condition (2.5) (for example $\mathbf{f} \equiv \boldsymbol{\varepsilon}$). Clearly for homogeneous boundary conditions $\boldsymbol{\varepsilon}^0(\mathbf{x}) \equiv \text{const.}$ (2.6) $\boldsymbol{\varepsilon}^{\Omega}(\mathbf{x})$ (2.8) is an invariant with respect to the cell number i and $\boldsymbol{\varepsilon}^{\Omega}(\mathbf{x}) = \langle \boldsymbol{\varepsilon} \rangle_{\mathbf{x}}(\mathbf{x}) = \text{const.}, \forall \mathbf{x} \in w$. In the general case of nonhomogeneous boundary conditions $\boldsymbol{\varepsilon}^0(\mathbf{x}) \neq \text{const.}$ (2.5) $\boldsymbol{\varepsilon}^{\Omega}(\mathbf{x})$ is a step function $\boldsymbol{\varepsilon}^{\Omega}(\mathbf{x}) \neq \boldsymbol{\varepsilon}^{\Omega}(\mathbf{y})$ at $\mathbf{x} \in \Omega_i$ and $\mathbf{y} \in \Omega_j$ $(i \neq j)$ as well as $\boldsymbol{\varepsilon}^{\Omega}(\mathbf{x}) \neq \langle \boldsymbol{\varepsilon} \rangle_{\mathbf{x}}(\mathbf{x})$ for $\mathbf{x} \in \Omega_i$.

3. General integral equations and effective field hypothesis

3.1. General integral equations

From eqns (2.1)–(2.4) a general integral equation for $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ can be derived. Substituting (2.3) and (2.1) into the equilibrium eqn (2.2), we obtain a differential equation with respect to the strain $\boldsymbol{\varepsilon}$

$$\nabla \mathbf{L}^{(0)} \boldsymbol{\varepsilon}(\mathbf{x}) = -\nabla \mathbf{L}_1(\mathbf{x}) \boldsymbol{\varepsilon}(\mathbf{x}), \tag{3.1}$$

which may be reduced to a symmetrized integral form

$$\boldsymbol{\varepsilon}(\mathbf{x}) = \boldsymbol{\varepsilon}^0(\mathbf{x}) - \nabla \int \mathbf{G}(\mathbf{x}-\mathbf{y}) \nabla [\mathbf{L}_1(\mathbf{y}) \boldsymbol{\varepsilon}(\mathbf{y})]\, d\mathbf{y}, \tag{3.2}$$

where $\boldsymbol{\varepsilon}^0(\mathbf{x})$ is the strain which would exist in the medium under the same boundary conditions if $\mathbf{L}$ was constant; $\mathbf{G}$ is the infinite-homogeneous-body Green's function of the Lamé equation with an elastic modulus tensor $\mathbf{L}^{(0)}$

$$\nabla \{ \mathbf{L}^{(0)} \tfrac{1}{2} [\nabla \otimes \mathbf{G}(\mathbf{x}) + (\nabla \otimes \mathbf{G}(\mathbf{x}))^{\mathrm{T}}] \} = -\boldsymbol{\delta} \delta(\mathbf{x}), \tag{3.3}$$

$\delta(\mathbf{x})$ is the Dirac delta function, $\boldsymbol{\delta}$ is the unit second-order tensor.

After integration of eqn (3.2) by parts, it is found that

$$\boldsymbol{\varepsilon}(\mathbf{x}) = \boldsymbol{\varepsilon}^0(\mathbf{x}) + \int \mathbf{U}(\mathbf{x}-\mathbf{y})\mathbf{L}_1(\mathbf{y})\boldsymbol{\varepsilon}(\mathbf{y})\,d\mathbf{y} + \oint \nabla\mathbf{G}(\mathbf{x}-\mathbf{s})\mathbf{L}_1(\mathbf{s})\boldsymbol{\varepsilon}(\mathbf{s})\mathbf{n}(\mathbf{s})\,d\mathbf{s}, \tag{3.4}$$

where the surface integration is taken over the boundary ∂w of the mesodomain w, containing a statistically large number of inclusions; $\mathbf{n}$ is the unit outward normal. The integral operator kernel $\mathbf{U}$ is defined by the Green tensor $\mathbf{G}$ (3.3):

$$U_{ijkl}(\mathbf{x}) = [\nabla_j \nabla_l G_{ik}(\mathbf{x})]_{(ij)(kl)}, \tag{3.5}$$

where the notation indicates symmetrization on (ij) and (kl).

Equation (3.4) is centered, i.e. from both sides of eqn (3.4) their average over the unit cell (2.9) are subtracted

$$\boldsymbol{\varepsilon}(\mathbf{x}) = \langle\boldsymbol{\varepsilon}\rangle_{\mathbf{x}}(\mathbf{x}) + \int \{\mathbf{U}(\mathbf{x}-\mathbf{y}) - \langle\mathbf{U}\rangle_{\mathbf{x}}(\mathbf{x}-\mathbf{y})\}\mathbf{L}_1(\mathbf{y})\boldsymbol{\varepsilon}(\mathbf{y})(\mathbf{y})\}\,d\mathbf{y} + \oint \{\nabla\mathbf{G}(\mathbf{x}-\mathbf{s}) - \nabla\langle\mathbf{G}\rangle_{\mathbf{x}}(\mathbf{x}-\mathbf{s})\}\mathbf{L}_1(\mathbf{s})\boldsymbol{\varepsilon}(\mathbf{s})\mathbf{n}(\mathbf{s})\,d\mathbf{s}, \tag{3.6}$$

where $\mathbf{x} \in \Omega_i$. For the analyses of integral convergence in eqn (3.6) we expand $\mathbf{U}(\mathbf{x}-\mathbf{y})$ in Taylor series about $\mathbf{x}_i^\Omega$ and integrate term by term over the unit cell Ω_i, then

$$\mathbf{U}(\mathbf{x}-\mathbf{y}) = \mathbf{U}(\mathbf{x}_i^\Omega-\mathbf{y}) + (\mathbf{x}-\mathbf{x}_i^\Omega)\nabla\mathbf{U}(\mathbf{x}_i^\Omega-\mathbf{y}) + \frac{1}{2}(\mathbf{x}-\mathbf{x}_i^\Omega)\otimes(\mathbf{x}-\mathbf{x}_i^\Omega)\nabla\nabla\mathbf{U}(\mathbf{x}_i^\Omega-\mathbf{y})\ldots,$$

$$\langle\mathbf{U}\rangle_{\mathbf{x}}(\mathbf{x}-\mathbf{y}) = \mathbf{U}(\mathbf{x}_i^\Omega-\mathbf{y}) + \frac{1}{2\bar{\Omega}_i}\int_{\Omega_i}(\mathbf{z}+\mathbf{x}-\mathbf{x}_i^\Omega)\otimes(\mathbf{z}+\mathbf{x}-\mathbf{x}_i^\Omega)\,d\mathbf{z}\nabla\nabla\mathbf{U}(\mathbf{x}_i^\Omega-\mathbf{y})\ldots. \tag{3.7}$$

Substituting (3.7) into eqn (3.6) shows that the term in curly brackets in the volume integral is of order $O(|\mathbf{x}-\mathbf{y}|^{-4})$; at $\mathbf{x} = \mathbf{x}_i^\Omega$ the indicated term is of order $O(|\mathbf{x}-\mathbf{y}|^{-5})$. At a sufficient distance $\mathbf{x}$ and $\mathbf{x}_i^\Omega$ from the boundary ∂w and $|\mathbf{x}-\mathbf{y}| \to \infty$ the integration over $\mathbf{y}$ can be carried out independently for both the expression in curly brackets $\{\mathbf{U}(\mathbf{x}-\mathbf{y}) - \langle\mathbf{U}\rangle_{\mathbf{x}}(\mathbf{x}-\mathbf{y})\}$ (3.6) (the function of the 'slow' variable $(\mathbf{x}-\mathbf{y})$) and the term $\mathbf{L}_1(\mathbf{y})\boldsymbol{\varepsilon}(\mathbf{y})$ (the function of 'fast' variable $\mathbf{y}$) and therefore, the volume integral in (3.6) converges absolutely. In a similar manner the term in curly brackets in the surface integral is of order either $O(|\mathbf{x}-\mathbf{y}|^{-3})$ (at $\mathbf{x} \neq \mathbf{x}_i^\Omega$) or $O(|\mathbf{x}-\mathbf{y}|^{-4})$ (at $\mathbf{x} = \mathbf{x}_i^\Omega$), and the surface integral vanishes at $|\mathbf{x}-\mathbf{s}| \to \infty$, $\mathbf{s} \in \partial w$. Moreover, in eqn (3.6) and below in the interest of obtaining explicit final expressions, we neglect by $\boldsymbol{\varepsilon}^0(\mathbf{x}) - \langle\boldsymbol{\varepsilon}^0\rangle_{\mathbf{x}}(\mathbf{x})$ as compared with $\langle\boldsymbol{\varepsilon}\rangle_{\mathbf{x}}(\mathbf{x})$ in the 'slowly-varying' approximation of $\boldsymbol{\varepsilon}^0(\mathbf{x})$.

By this means eqn (3.6) is reduced to the relation

$$\boldsymbol{\varepsilon}(\mathbf{x}) = \langle\boldsymbol{\varepsilon}\rangle_{\mathbf{x}}(\mathbf{x}) + \int \{\mathbf{U}(\mathbf{x}-\mathbf{y}) - \langle\mathbf{U}\rangle_{\mathbf{x}}(\mathbf{x}-\mathbf{y})\}\mathbf{L}_1(\mathbf{y})\boldsymbol{\varepsilon}(\mathbf{y})\,d\mathbf{y}, \tag{3.8}$$

where the volume integral converges absolutely. The principal advantages of eqn (3.8) as compared with the equivalent eqn (3.6) are the lack of the surface integral in eqn (3.8) and the local character of eqn (3.8). The last-mentioned advantage makes it possible to reduce the analyses of infinite number inclusion problems to the analysis of a finite number of inclusions located in some RVE (see Section 6 for details).

It should be mentioned that for triply periodic structures at the uniform boundary conditions $\boldsymbol{\varepsilon}^0(\mathbf{x}) \equiv \boldsymbol{\varepsilon}^0 = \text{const.}$ (2.6) the term $\mathbf{L}_1(\mathbf{y})\boldsymbol{\varepsilon}(\mathbf{y})$ (3.8) is an invariant with respect to the unit cell number and eqn (3.8) can be rewritten in the form

$$\boldsymbol{\varepsilon}(\mathbf{x}) = \boldsymbol{\varepsilon}^{\Omega}(\mathbf{x}) + \int \mathbf{U}(\mathbf{x}-\mathbf{y})\{\mathbf{L}_1(\mathbf{y})\boldsymbol{\varepsilon}(\mathbf{y}) - (\mathbf{L}_1\boldsymbol{\varepsilon})^{\Omega}(\mathbf{x})\}\, d\mathbf{y}, \tag{3.9}$$

which was previously used by Buryachenko and Parton (1992) for both $(\mathbf{L}_1\boldsymbol{\varepsilon})^{\Omega}(\mathbf{x}) \equiv$ const. and $\langle\boldsymbol{\varepsilon}\rangle_{\mathbf{x}}(\mathbf{x}) = \boldsymbol{\varepsilon}^{\Omega}(\mathbf{x}) \equiv$ const.

Obtained general integral eqns (3.8) and (3.9) are valid not only for deterministic structures but can be generalized also for a statistically homogeneous ergodic inclusion field as well as for statistically inhomogeneous structures. In the case of random structure composites let $\langle(\cdot)\rangle^{\mathrm{st}}(\mathbf{x})$ denote the statistical average for the ensemble of a field X and one reduces eqn (3.6) to (see Buryachenko and Parton, 1990)

$$\boldsymbol{\varepsilon}(\mathbf{x}) = \langle\boldsymbol{\varepsilon}\rangle^{\mathrm{st}}(\mathbf{x}) + \int \mathbf{U}(\mathbf{x}-\mathbf{y})\{\mathbf{L}_1(\mathbf{y})\boldsymbol{\varepsilon}(\mathbf{y}) - \langle\mathbf{L}_1\boldsymbol{\varepsilon}\rangle^{\mathrm{st}}(\mathbf{y})\}\, d\mathbf{y}, \quad (\mathbf{x}\in v_i), \tag{3.10}$$

as alternatives to eqn (3.8). In eqn (3.10) it is taken into account that, at sufficient distance $\mathbf{x}$ from the boundary ∂w, the operation of surface integration may be regarded as statistical averaging (see e.g. Shermergor, 1977).

For a statistically homogeneous ergodic field X [as well as for the field (2.7)] the integral eqn (3.10) is equivalent to those known from the literature (Levin, 1976; Kröner, 1977; O'Brian, 1979; Willis, 1982; Buryachenko and Lipanov, 1986). Nevertheless, for statistically inhomogeneous field X [as well as for inhomogeneous fields either $\langle\boldsymbol{\varepsilon}\rangle_{\mathbf{x}}(\mathbf{x})$ (3.8) or $\langle\boldsymbol{\varepsilon}\rangle^{\mathrm{st}}(\mathbf{x})$ (3.10)] the dependence of statistical averages $\langle(\cdot)\rangle(\mathbf{y})$ (3.10) of the current coordinate $\mathbf{y}$ is of fundamental importance. But even in this case the expression in curly brackets eqns (3.10) (which is a perturbation introduced by the inclusion v_i at the point $\mathbf{y}$) is of order $O(|\mathbf{x}-\mathbf{y}|^{-3})$ as $|\mathbf{x}-\mathbf{y}| \to \infty$ and the integral in eqns (3.10) converges absolutely. Therefore, there are no difficulties connected with the asymptotic behavior at the infinity (as $|\mathbf{x}-\mathbf{y}|^{-3}$) of the generalized functions $\mathbf{U}$, and there is no need to postulate the form of the domain w or to resort to regularization of integrals (see e.g. Kunin, 1983; Kröner, 1990; Ju and Chen, 1994) which are divergent at infinity and which is difficult in the case of statistical inhomogeneity of the field X [as well as in the cases of inhomogeneous fields either $\langle\boldsymbol{\varepsilon}\rangle_{\mathbf{x}}(\mathbf{x})$ (3.8) or $\langle\boldsymbol{\varepsilon}\rangle^{\mathrm{st}}(\mathbf{x})$ (3.10)]. Equations (3.8)–(3.10) are the theoretical basis of the locality principle by Sokolkin and Tashkinov (1984), which uses the short-range-order effect in the interactions of the periodic or random problem in the boundary problem for the RVE with a finite number of inhomogeneities. The locality principle was justified by Buryachenko and Parton (1992) for the particular case of simple cubic packing of the spherical inclusions; they showed that the maximum error in using the representative volume (if the RVE dimensions are more than three times the inclusion spacing) is smaller than 2%.

3.2. Approximative effective field hypothesis

Now we define the effective field $\mathfrak{E}_i(\mathbf{x})$ $(\mathbf{x}\in v_i, i = 1,\ldots)$ (generally speaking nonhomogeneous) as a strain field in which the chosen inclusion v_i is embedded:

$$\mathfrak{E}(\mathbf{x}) = \langle\boldsymbol{\varepsilon}\rangle_{\mathbf{x}}(\mathbf{x}) + \int [\mathbf{U}(\mathbf{x}-\mathbf{y})\mathbf{L}_1(\mathbf{y})\boldsymbol{\varepsilon}(\mathbf{y})(1-V_i(\mathbf{y})) - \langle\mathbf{U}\rangle_{\mathbf{x}}(\mathbf{x}-\mathbf{y})\mathbf{L}_1(\mathbf{y})\boldsymbol{\varepsilon}(\mathbf{y})]\, d\mathbf{y}, \quad (\mathbf{x}\in v_i). \tag{3.11}$$

In order to simplify the system (3.8) we now apply the main hypothesis of many micromechanical methods, the so-called effective field hypothesis:

(H1) Each inclusion v_i has an ellipsoidal form and is located in the field $\mathfrak{E}_i \equiv \mathfrak{E}(\mathbf{x})(\mathbf{x}\in v_i)$ which is

homogeneous over the inclusion v_i. The perturbations introduced by the inclusion v_j in the point $\mathbf{x} \notin v_j$ are defined by the relations

$$\int \mathbf{U}(\mathbf{x}-\mathbf{y}) V_j(\mathbf{y}) \mathbf{L}_1(\mathbf{y}) \boldsymbol{\varepsilon}(\mathbf{y}) \, d\mathbf{y} = \mathbf{T}_j(\mathbf{x}-\mathbf{x}_j) \bar{\boldsymbol{\varepsilon}}(\mathbf{x}_j),$$
$$\int \langle \mathbf{U} \rangle_{\mathbf{x}}(\mathbf{x}-\mathbf{y}) V_j(\mathbf{y}) \mathbf{L}_1(\mathbf{y}) \boldsymbol{\varepsilon}(\mathbf{y}) \, d\mathbf{y} = \langle \mathbf{T}_j \rangle_{\mathbf{x}}(\mathbf{x}-\mathbf{x}_j) \bar{\boldsymbol{\varepsilon}}(\mathbf{x}_j). \tag{3.12}$$

Here one introduces the tensors

$$\mathbf{T}_j(\mathbf{x}-\mathbf{x}_j) = \begin{cases} \bar{v}_j^{-1} \int \mathbf{U}(\mathbf{x}-\mathbf{y}) V_j(\mathbf{y}) \, d\mathbf{y} \mathbf{R} & \text{for } \mathbf{x} \notin v_j, \\ -\bar{v}_j^{-1} \mathbf{P} \mathbf{R} & \text{for } \mathbf{x} \in v_j, \end{cases}$$

$$\langle \mathbf{T}_j \rangle_{\mathbf{x}}(\mathbf{x}-\mathbf{x}_j) = \begin{cases} \bar{v}_j^{-1} \int \langle \mathbf{U} \rangle_{\mathbf{x}}(\mathbf{x}-\mathbf{y}) V_j(\mathbf{y}) \, d\mathbf{y} \mathbf{R} & \text{for } \mathbf{x} \notin v_j, \\ -\bar{v}_j^{-1} \mathbf{P} \mathbf{R} & \text{for } \mathbf{x} \in v_j, \end{cases} \tag{3.13}$$

where the tensor $\mathbf{P}$ is associated with the well-known Eshelby tensor by

$$\mathbf{S} = \mathbf{P} \mathbf{L}^{(0)}, \quad \mathbf{S} \equiv -\int \mathbf{U}(\mathbf{x}-\mathbf{y}) V_j(\mathbf{y}) \, d\mathbf{y} = \text{const.}, \quad (\mathbf{x}, \mathbf{y} \in v_j), \tag{3.14}$$

and the tensor $\mathbf{R} = \text{const.}$ is defined later.

Then in the framework of the hypotheses (H1) and in view of the linearity of the problem there exist constant fourth-rank tensors $\mathbf{A}(\mathbf{x})$, $\mathbf{R}(\mathbf{x})$, such that

$$\boldsymbol{\varepsilon}(\mathbf{x}) = \mathbf{A}(\mathbf{x}) \bar{\boldsymbol{\varepsilon}}(\mathbf{x}), \quad \bar{v}_i \mathbf{L}_1(\mathbf{x}) \boldsymbol{\varepsilon}(\mathbf{x}) = \mathbf{R}(\mathbf{x}) \bar{\boldsymbol{\varepsilon}}(\mathbf{x}), \quad \mathbf{x} \in v_i, \tag{3.15}$$

where $\mathbf{R}(\mathbf{x}) \equiv \bar{v}_i \mathbf{L}_1(\mathbf{x}) \mathbf{A}(\mathbf{x})$. According to Eshelby's (1961) theorem there are following relations between the averaged tensors (3.15)

$$\mathbf{R} = \bar{v}_i \mathbf{P}^{-1} (\mathbf{I} - \mathbf{A}), \tag{3.16}$$

where

$$\mathbf{f} \equiv \langle \mathbf{f}(\mathbf{x}) \rangle_{(i)} = \bar{v}_i^{-1} \int \mathbf{f}(\mathbf{x}) V_i(\mathbf{x}) \, d\mathbf{x}$$

denotes averaging over the volume of the inclusion v_i and $\mathbf{f}$ stands for $\mathbf{R}$ (3.9) and for $\mathbf{A}$.

For example, for the homogeneous ellipsoidal domain v_i with

$$\mathbf{L}_1(\mathbf{x}) = \mathbf{L}_1^{(1)} = \text{const}, \quad \text{at } \mathbf{x} \in v_i, \tag{3.17}$$

we get

$$\mathbf{A} = (\mathbf{I} + \mathbf{P} \mathbf{L}_1^{(1)})^{-1}, \quad \mathbf{R} = \bar{v}_i \mathbf{L}_1^{(1)} \mathbf{A}. \tag{3.18}$$

From comparison between the relations (3.15) and (3.18) we see that the average elastic response (i.e. the tensors $\mathbf{A}$, $\mathbf{R}$) of any coated inclusion v_i is the same as the response of some fictitious ellipsoidal homogeneous inclusion with elastic moduli

$$\mathbf{L}_1^{\mathrm{f}(1)} = \mathbf{P}^{-1} (\mathbf{A}^{-1} - \mathbf{I}), \tag{3.19}$$

which can be expressed in terms of the tensor $\mathbf{R}$:

$$\mathbf{L}_1^{\mathrm{f}(1)} = (\bar{v}_i \mathbf{I} - \mathbf{R} \mathbf{P})^{-1} \mathbf{R}. \tag{3.20}$$

The parameters (3.19) and (3.20) of fictitious ellipsoidal inclusions are simply a notational convenience. No restrictions are imposed on the microtopology of the coated inclusions as well as on the inhomogeneity of the stress state in the coated inclusions (see for details Buryachenko and Rammerstorfer, 1998).

Strictly speaking the hypothesis (H1) cannot be satisfied, nevertheless Buryachenko and Parton (1990) showed that using it provides a high precision analysis of some regular structures under the homogeneous external loading. Because of this we will employ the hypothesis (H1).

4. Estimation of effective strains in the inclusions

4.1. The Fourier transform method

In the framework of the hypothesis (H1) the system (3.8) for the periodic structure is reduced to

$$\bar{\boldsymbol{\varepsilon}}(\mathbf{x}_i) = \langle \boldsymbol{\varepsilon} \rangle_{\mathbf{x}_i}(\mathbf{x}_i) + \int \mathscr{F}(\mathbf{x}_i - \mathbf{y}) \bar{\boldsymbol{\varepsilon}}(\mathbf{y})\, d\mathbf{y}, \tag{4.1}$$

the tensor $\mathscr{F}(\mathbf{x}_i - \mathbf{y})$ introduced in eqn (4.1) is defined as

$$\mathscr{F}(\mathbf{x}_i - \mathbf{y}) = \sum_{\mathbf{m}} [\mathbf{T}_{i\mathbf{m}}(\mathbf{x}_i - \mathbf{x}_\mathbf{m})(1 - V_i(\mathbf{y})) - \langle \mathbf{T}_\mathbf{m} \rangle_{\mathbf{x}_i}(\mathbf{x}_i - \mathbf{x}_\mathbf{m})] \delta(\mathbf{y} - \mathbf{x}_\mathbf{m}), \tag{4.2}$$

and

$$\mathbf{T}_{i\mathbf{m}}(\mathbf{x}_i - \mathbf{x}_\mathbf{m}) = (\bar{v}_i \bar{v}_\mathbf{m})^{-1} \int \mathbf{U}(\mathbf{x} - \mathbf{y}) V_i(\mathbf{x}) V_\mathbf{m}(\mathbf{y})\, d\mathbf{x}\, d\mathbf{y}\mathbf{R}. \tag{4.3}$$

According to the definition (3.10) the fields $\bar{\boldsymbol{\eta}}(\mathbf{x}_i)$ are invariants with respect to the inclusion number i and depend on the argument $\mathbf{x}$. Since we desire an explicit representation for $\bar{\boldsymbol{\varepsilon}}(\mathbf{x})$ we will approximate $\bar{\boldsymbol{\varepsilon}}(\mathbf{y})$ by the first three terms of its Tailor expansion about $\mathbf{x}$:

$$\bar{\boldsymbol{\varepsilon}}(\mathbf{y}) \approx \bar{\boldsymbol{\varepsilon}}(\mathbf{x}) + (\mathbf{y} - \mathbf{x}) \nabla \bar{\boldsymbol{\varepsilon}}(\mathbf{x}) + \tfrac{1}{2}(\mathbf{y} - \mathbf{x}) \otimes (\mathbf{y} - \mathbf{x}) \nabla \nabla \bar{\boldsymbol{\varepsilon}}(\mathbf{x}). \tag{4.4}$$

Substituting (4.4) into eqn (4.1) gives

$$\bar{\boldsymbol{\varepsilon}}(\mathbf{x}_i) = \langle \boldsymbol{\varepsilon} \rangle_{\mathbf{x}_i}(\mathbf{x}_i) + \int \mathscr{F}(\mathbf{x}_i - \mathbf{y})\, d\mathbf{y} \bar{\boldsymbol{\varepsilon}}(\mathbf{x}_i) + \int \mathscr{F}_1(\mathbf{x}_i - \mathbf{y})\, d\mathbf{y} \nabla \bar{\boldsymbol{\varepsilon}}(\mathbf{x}_i) + \int \mathscr{F}_2(\mathbf{x}_i - \mathbf{y})\, d\mathbf{y} \nabla \nabla \bar{\boldsymbol{\varepsilon}}(\mathbf{x}_i), \tag{4.5}$$

where the integral operator kernels $\mathscr{F}_1(\mathbf{x}_i - \mathbf{y})$ and $\mathscr{F}_2(\mathbf{x}_i - \mathbf{y})$ are defined by the tensor $\mathscr{F}(\mathbf{x}_i - \mathbf{y})$ (4.2):

$$\begin{aligned} \mathscr{F}_1(\mathbf{x}_i - \mathbf{y}) &= \mathscr{F}(\mathbf{x}_i - \mathbf{y}) \otimes (\mathbf{y} - \mathbf{x}_i), \\ \mathscr{F}_2(\mathbf{x}_i - \mathbf{y}) &= \tfrac{1}{2} \mathscr{F}(\mathbf{x}_i - \mathbf{y}) \otimes (\mathbf{y} - \mathbf{x}_i) \otimes (\mathbf{y} - \mathbf{x}_i). \end{aligned} \tag{4.6}$$

For triply periodic structures all integrals in eqn (4.5) are constant tensors. Moreover, the second right-hand-side integral in eqn (4.5) vanishes, because the tensor $\mathscr{F}_1(\mathbf{x}_i - \mathbf{y})$ is an odd function: $\mathscr{F}_1(\mathbf{x}_i - \mathbf{y}) = -\mathscr{F}_1(\mathbf{y} - \mathbf{x}_i)$.

Considering that eqn (4.5) is a differential equation with constant coefficients, the method of solution that first comes to mind is using of Fourier transform to transform the differential problem

of solving (4.5) into the division problem of solving the multiplicative equation (see e.g. Treves, 1980)

$$\mathscr{P}(i\boldsymbol{\xi})\tilde{\boldsymbol{\varepsilon}}(\boldsymbol{\xi}) = \langle\tilde{\boldsymbol{\varepsilon}}\rangle(\boldsymbol{\xi}), \tag{4.7}$$

where a symbol

$$\mathscr{P}(i\boldsymbol{\xi}) = \mathbf{I} - \int [\mathscr{F}(\mathbf{x}_i - \mathbf{y}) - \mathscr{F}_2(\mathbf{x}_i - \mathbf{y})\boldsymbol{\xi} \otimes \boldsymbol{\xi}]\, d\mathbf{y} \tag{4.8}$$

of the differential operator (4.6) is a polynomial with real constant coefficients in three real transform variable $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)^{\mathrm{T}}$. Here the Fourier transformation $\tilde{\mathbf{g}}(\boldsymbol{\xi})$ of a function $\mathbf{g}(\mathbf{x})$ and its inverse are defined by the formulae

$$F(\mathbf{g}) \equiv \tilde{\mathbf{g}}(\boldsymbol{\xi}) = \int \mathbf{g}(\mathbf{x})\, e^{-i\boldsymbol{\xi}\cdot\mathbf{x}}\, d\mathbf{x}, \quad \mathbf{g}(\mathbf{x}) = F^{-1}(\tilde{\mathbf{g}}) = \frac{1}{8\pi^3}\int \tilde{\mathbf{g}}(\boldsymbol{\xi})\, e^{i\boldsymbol{\xi}\cdot\mathbf{x}}\, d\boldsymbol{\xi} \tag{4.9}$$

provided, of course, that the integrals on the right-hand sides of the equations are convergent.

Therefore,

$$\boldsymbol{\varepsilon}(\mathbf{x}) = \frac{1}{8\pi^3}\int e^{i\boldsymbol{\xi}\cdot\mathbf{x}}\mathscr{P}^{-1}(i\boldsymbol{\xi})\langle\tilde{\boldsymbol{\varepsilon}}\rangle(\boldsymbol{\xi})\, d\boldsymbol{\xi} \tag{4.10}$$

should be a solution of (4.6) in view of (4.9). Substituting (4.8) into (4.10) and restricting the result to terms of no greater than second-order in the expansion

$$\mathscr{P}^{-1}(i\boldsymbol{\xi}) \approx \mathbf{Y} - \mathscr{Y} \otimes \boldsymbol{\xi} \otimes \boldsymbol{\xi}, \tag{4.11}$$

we get

$$\boldsymbol{\varepsilon}(\mathbf{x}_i) = \mathbf{Y}\langle\boldsymbol{\varepsilon}\rangle_{\mathbf{x}_i}(\mathbf{x}_i) + \mathscr{Y}\nabla\nabla\langle\boldsymbol{\varepsilon}\rangle_{\mathbf{x}_i}(\mathbf{x}_i), \tag{4.12}$$

where the local part $\mathbf{Y}$ of the differential operator (4.12) has the form

$$\mathbf{Y} = [\mathbf{I} - \int \mathscr{F}(\mathbf{x}_i - \mathbf{y})\, d\mathbf{y}]^{-1}, \tag{4.13}$$

and the nonlocal part of the differential operator (4.12) is defined by the tensor

$$\mathscr{Y} = \mathbf{Y}\int \mathscr{F}_2(\mathbf{x}_i - \mathbf{y})\, d\mathbf{y}\mathbf{Y}. \tag{4.14}$$

To summarize, we have found a nonlocal effective field representation for triply periodic composites having periodic distribution of arbitrarily-shape coated inclusions and arbitrarily anisotropic phases.

4.2. Iteration method

Strictly speaking eqn (4.12) is defined at both sufficiently slowly-varying fields $\langle\boldsymbol{\varepsilon}\rangle_{\mathbf{x}}(\mathbf{x})$ and $\bar{\boldsymbol{\varepsilon}}(\mathbf{x})$, and should be considered as an approximation of the real nonlocal operator (4.10) by the differential operator of the second-order. For elimination of these limitations we will represent the right-hand-side of eqn (4.12) in the form of the integral operator. With this aim it should be mentioned that in addition to the method of Fourier transforms used above there are other approximate methods which can be employed for solving Fredholm integral equations of the second kind such

as eqn (4.1) (see e.g. Delves and Mohamed, 1985). For instance, if the kernel of a Fredholm integral equation of the second kind is small enough then the answer for $\bar{\boldsymbol{\varepsilon}}(\mathbf{x}_i)$ can be constructed by the method of Liouville–Neumann series proceeding by the recurrence formula

$$\bar{\boldsymbol{\varepsilon}}(\mathbf{x}_i)^{(n+1)} = \langle\boldsymbol{\varepsilon}\rangle_{\mathbf{x}_i}(\mathbf{x}_i) + \int \mathscr{F}(\mathbf{x}_i - \mathbf{y})\bar{\boldsymbol{\varepsilon}}(\mathbf{y})^{(n)}\,\mathrm{d}\mathbf{y}. \tag{4.15}$$

Usually the driving term of this equation is used as an initial approximation: $\bar{\boldsymbol{\varepsilon}}(\mathbf{x}_i)^{(0)} = \langle\boldsymbol{\varepsilon}\rangle_{\mathbf{x}_i}(\mathbf{x}_i)$. However, let us assume a sufficiently slowly-varying average field $\langle\boldsymbol{\varepsilon}\rangle_{\mathbf{x}_i}(\mathbf{x}_i)$, and for $\langle\boldsymbol{\varepsilon}\rangle_{\mathbf{x}_i}(\mathbf{x}_i) \equiv$ const. it is known the exact solution of eqn (4.15) is given by

$$\bar{\boldsymbol{\varepsilon}}(\mathbf{x}_i) = \mathbf{Y}\langle\boldsymbol{\varepsilon}\rangle_{\mathbf{x}_i}(\mathbf{x}_i) \equiv \text{const.} \tag{4.16}$$

Then for slowly-varying average field $\langle\boldsymbol{\varepsilon}\rangle_{\mathbf{x}_i}(\mathbf{x}_i)$ it would appear reasonable to employ the value $\bar{\boldsymbol{\varepsilon}}(\mathbf{x}_i)^{(0)} = \mathbf{Y}\langle\boldsymbol{\varepsilon}\rangle_{\mathbf{x}_i}(\mathbf{x}_i) \neq$ const. as the zero-order approximation, so that the first-order approximation can be obtained from the modified recurrence relation (4.15) as follows

$$\boldsymbol{\varepsilon}(\mathbf{x}_i) = \mathbf{Y}\langle\boldsymbol{\varepsilon}\rangle_{\mathbf{x}_i}(\mathbf{x}_i) + \int \mathscr{L}(\mathbf{x}_i - \mathbf{y})[\langle\boldsymbol{\varepsilon}\rangle_{\mathbf{y}}(\mathbf{y}) - \langle\boldsymbol{\varepsilon}\rangle_{\mathbf{x}_i}(\mathbf{x}_i)]\,\mathrm{d}\mathbf{y}, \tag{4.17}$$

where the integral operator kernel $\mathscr{L}(\mathbf{x}_i - \mathbf{y})$ (4.17) is defined by the relation

$$\mathscr{L}(\mathbf{x}_i - \mathbf{y}) = \mathbf{Y}\mathscr{F}(\mathbf{x}_i - \mathbf{y})\mathbf{Y}. \tag{4.18}$$

It is obvious that for the homogeneous boundary conditions $\boldsymbol{\varepsilon}^0(\mathbf{x}) \equiv$ const. (2.5) carries into $\langle\boldsymbol{\varepsilon}\rangle_{\mathbf{y}}(\mathbf{y}) \equiv$ const., the right-hand-side integral in eqn (4.17) vanishes, and the solutions (4.16) and (4.17) coincide.

Now we prove that for sufficiently smooth average strain fields and some additional assumptions both the Fourier transform method and the iteration method lead to the same results. Really, it has been assumed previously that the average field $\langle\boldsymbol{\varepsilon}\rangle_{\mathbf{y}}(\mathbf{y})$ is slowly-varying enough. Then we can approximate $\langle\boldsymbol{\varepsilon}\rangle(\mathbf{y})$ by the first three terms of its Taylor expansion about $\mathbf{x}_i$:

$$\langle\boldsymbol{\varepsilon}\rangle_{\mathbf{y}}(\mathbf{y}) \approx \langle\boldsymbol{\varepsilon}\rangle_{\mathbf{x}_i}(\mathbf{x}_i) + (\mathbf{y} - \mathbf{x}_i)\nabla\langle\boldsymbol{\varepsilon}\rangle_{\mathbf{x}_i}(\mathbf{x}_i) + \tfrac{1}{2}(\mathbf{y} - \mathbf{x}_i) \otimes (\mathbf{y} - \mathbf{x}_i)\nabla\nabla\langle\boldsymbol{\varepsilon}\rangle_{\mathbf{x}_i}(\mathbf{x}_i). \tag{4.19}$$

Substituting (4.19) into (4.17) and since $\mathscr{L}$ is an even function (4.18), (4.17) finally reduces to (4.12). By this means for sufficiently slowly-varying average fields $\langle\boldsymbol{\varepsilon}\rangle_{\mathbf{y}}(\mathbf{y})$ the first few steps of both successive iterations and Taylor expansions in the iteration method and in the Fourier transform method, respectively, lead to the same relation (4.12). Nevertheless, since the Fourier transform method employed Taylor's expansion twice (4.4) and (4.11) one should expect that the iteration method is a better choice between these two methods.

The method of Fourier transform used above (4.12) has been investigated in non-local micromechanics of random structure composites and was used with the slight modifications by Beran and McCoy (1970), Buryachenko and Lipanov (1992), Drugan and Willis (1996), Khoroshun (1996), Buryachenko (1998). Nevertheless the differential nonlocal relation (4.12) has the disadvantage that it uses the concrete polynomial approximations (4.4) and (4.11) in a sufficiently large neighbourhood, which is sometime violented in practice. In contrast, the integral nonlocal eqn (4.17) does not use the concrete representations (4.4) and (4.7), and can be applied with the controlled accuracy for the analysis of a more wide class of average fields $\langle\boldsymbol{\varepsilon}\rangle_{\mathbf{y}}(\mathbf{y})$, since integration is a smoothing operation and the right-hand-side integral (4.17) is likely to be a rather smooth function even when $\langle\boldsymbol{\varepsilon}\rangle_{\mathbf{y}}(\mathbf{y})$ is very jagged. However, more detailed consideration of convergence

rates and an estimation of its accuracy for the procedures of nonlocal operators used here are beyond the scope of the current paper (this line of research will be pursued in the forthcoming paper by Buryachenko, 1999).

5. Average strains in the components and effective elastic properties

5.1. Average strains in the components

The strain field inside the inclusions $\boldsymbol{\varepsilon}(\mathbf{z})$ ($\mathbf{z}\in v_i$) is obtained from (3.15) and (4.12)

$$\boldsymbol{\varepsilon}(\mathbf{x}_i,\mathbf{z}) = \mathbf{A}(\mathbf{z})\mathbf{Y}\langle\boldsymbol{\varepsilon}\rangle_{\mathbf{x}_i}(\mathbf{x}_i)+\mathbf{A}(\mathbf{z})\mathscr{Y}\nabla\nabla\langle\boldsymbol{\varepsilon}\rangle_{\mathbf{x}_i}(\mathbf{x}_i), \tag{5.1}$$

from which the representation for the average strains inside the inclusion v_i follows

$$\langle\boldsymbol{\varepsilon}\rangle_{(i)} = \mathbf{A}\mathbf{Y}\langle\boldsymbol{\varepsilon}\rangle_{\mathbf{x}_i}(\mathbf{x}_i)+\mathbf{A}\mathscr{Y}\nabla\nabla\langle\boldsymbol{\varepsilon}\rangle_{\mathbf{x}_i}(\mathbf{x}_i), \tag{5.2}$$

here the 'fast' independent variable $\mathbf{z}\in v_i$ characterizing the strain state is defined in local coordinate system connected with the semiaxes of the ellipsoid v_i. There is connection between the 'slow' $\mathbf{x}$ and 'fast' $\mathbf{z}\in v_i$ variables: $\mathbf{x} = \Sigma m_j\mathbf{e}_j+\mathbf{z}$.

The mean matrix strains follow simply from eqn (5.2) and the relation

$$\langle\boldsymbol{\varepsilon}\rangle_0(\mathbf{x}) = \frac{1}{c^{(0)}}(\langle\boldsymbol{\varepsilon}\rangle_{\mathbf{x}}(\mathbf{x})-c^{(1)}\langle\boldsymbol{\varepsilon}\rangle_{(i)}), \tag{5.3}$$

where $\mathbf{x}\in\Omega_i\backslash v_i$. Substituting (5.2) into (3.8) gives the local strains in the matrix $\mathbf{z}\in\Omega_i\backslash v_i$ in the form of an integro-differential equation

$$\boldsymbol{\varepsilon}(\mathbf{x}_i,\mathbf{z}) = \langle\boldsymbol{\varepsilon}\rangle_{\mathbf{x}_i}(\mathbf{x}_i)+\int\sum_{\mathbf{m}}[\mathbf{T}_{\mathbf{m}}(\mathbf{z}-\mathbf{y})-\langle\mathbf{T}\rangle_{\mathbf{z}}(\mathbf{z}-\mathbf{y})]\delta(\mathbf{y}-\mathbf{x}_{\mathbf{m}})\cdot[\mathbf{Y}\langle\boldsymbol{\varepsilon}\rangle_{\mathbf{y}}(\mathbf{y})+\mathscr{Y}\nabla\nabla\langle\boldsymbol{\varepsilon}\rangle_{\mathbf{y}}]\,\mathrm{d}\mathbf{y}. \tag{5.4}$$

When using the integral eqn (4.17) rather than differential dependence form of the effective strain $\tilde{\boldsymbol{\varepsilon}}(\mathbf{x}_i)$ on the mean strain $\boldsymbol{\varepsilon}^{\Omega}(\mathbf{x}_i)$, eqns (5.1), (5.2) and (5.4) should be replaced by

$$\begin{aligned}
\boldsymbol{\varepsilon}(\mathbf{x}_i,\mathbf{z}) &= \mathbf{A}(\mathbf{z})\{\mathbf{Y}\langle\boldsymbol{\varepsilon}\rangle_{\mathbf{x}_i}(\mathbf{x}_i)+\int\mathscr{Z}(\mathbf{x}_i-\mathbf{y})[\langle\boldsymbol{\varepsilon}\rangle_{\mathbf{y}}(\mathbf{y})-\langle\boldsymbol{\varepsilon}\rangle_{\mathbf{x}_i}(\mathbf{x}_i)]\,\mathrm{d}\mathbf{y}\},\\
\langle\boldsymbol{\varepsilon}\rangle_{(i)} &= \mathbf{A}\{\mathbf{Y}\langle\boldsymbol{\varepsilon}\rangle_{\mathbf{x}_i}(\mathbf{x}_i)+\int\mathscr{Z}(\mathbf{x}_i-\mathbf{y})[\langle\boldsymbol{\varepsilon}\rangle_{\mathbf{y}}(\mathbf{y})-\langle\boldsymbol{\varepsilon}\rangle_{\mathbf{x}_i}(\mathbf{x}_i)]\,\mathrm{d}\mathbf{y}\},\\
\boldsymbol{\varepsilon}(\mathbf{x}_i,\mathbf{z}) &= \langle\boldsymbol{\varepsilon}\rangle_{\mathbf{x}_i}(\mathbf{x}_i)+\int\sum_{\mathbf{m}}[\mathbf{T}_{\mathbf{m}}(\mathbf{z}-\mathbf{y})-\langle\mathbf{T}_{\mathbf{z}}\rangle_{\mathbf{z}}(\mathbf{z}-\mathbf{y})]\delta(\mathbf{y}-\mathbf{x}_{\mathbf{m}})\\
&\quad\cdot\{\mathbf{Y}\langle\boldsymbol{\varepsilon}\rangle_{\mathbf{x}_{\mathbf{m}}}(\mathbf{x}_{\mathbf{m}})+\int\mathscr{Z}(\mathbf{x}_{\mathbf{m}}-\mathbf{t})[\langle\boldsymbol{\varepsilon}\rangle_{\mathbf{t}}(\mathbf{t})-\langle\boldsymbol{\varepsilon}\rangle_{\mathbf{x}_{\mathbf{m}}}(\mathbf{x}_{\mathbf{m}})]\,\mathrm{d}\mathbf{t}\}\,\mathrm{d}\mathbf{y},
\end{aligned} \tag{5.5}$$

respectively.

5.2. Effective properties of composites

Taking the average strain in the inclusions (5.2) gives a macroscopic constitutive equation that relates $\langle\boldsymbol{\sigma}\rangle(\mathbf{x})$ and $\langle\boldsymbol{\varepsilon}\rangle(\mathbf{x})$:

$$\langle\boldsymbol{\sigma}\rangle(\mathbf{x}) = \mathbf{L}^*\langle\boldsymbol{\varepsilon}\rangle(\mathbf{x})+\mathscr{L}^*\nabla\nabla\langle\boldsymbol{\varepsilon}\rangle(\mathbf{x}), \tag{5.6}$$

$$\mathbf{L}^* = \mathbf{L}^{(0)} + \mathbf{R}\mathbf{Y}n, \tag{5.7}$$

$$\mathscr{L}^* = \mathbf{R}\mathscr{Y}n. \tag{5.8}$$

The treatment of the integral form of the differential macroscopic constitutive eqn (5.6) leads to the nonlocal relation

$$\langle\boldsymbol{\sigma}\rangle(\mathbf{x}) = \mathbf{L}^*\langle\boldsymbol{\varepsilon}\rangle(\mathbf{x}) + n\mathbf{R}\int \mathscr{Z}(\mathbf{x}-\mathbf{y})[\langle\boldsymbol{\varepsilon}\rangle_{\mathbf{y}}(\mathbf{y}) - \langle\boldsymbol{\varepsilon}\rangle_{\mathbf{x}}(\mathbf{x})]\,\mathrm{d}\mathbf{y}. \tag{5.9}$$

Let for the sake of definiteness the composite material is subjected to a strain gradient along the direction $\mathbf{e}_3$ of the orthogonal basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$. Then

$$\langle\boldsymbol{\varepsilon}\rangle_{\mathbf{x}}(\mathbf{x}) = f(x_3)\boldsymbol{\varepsilon}^{\mathrm{con}}, \quad \mathbf{x} = (x_1, x_2, x_3)^{\mathrm{T}}, \tag{5.10}$$

where $\boldsymbol{\varepsilon}^{\mathrm{con}} \equiv$ const. and $f(x_3) \neq$ const., and a three-dimensional integral (5.9) can be reduced to the one-dimensional

$$\langle\boldsymbol{\sigma}\rangle_{\mathbf{x}}(\mathbf{x}) = \mathbf{L}^*\boldsymbol{\varepsilon}^{\mathrm{con}} f(x_3) + \int \mathscr{Z}^1(x_3 - y_3)[f(y_3) - f(x_3)]\,\mathrm{d}y_3\,\boldsymbol{\varepsilon}^{\mathrm{con}}, \tag{5.11}$$

where the integral operator kernel is defined by the relation

$$\mathscr{Z}^1(x_3 - y_3) = \mathrm{n}\mathbf{R}\int\int \mathscr{Z}(\mathbf{x}-\mathbf{y})\,\mathrm{d}y_1\,\mathrm{d}y_2. \tag{5.12}$$

The preceding relation for the local effective properties $\mathbf{L}^*$ may be simplified by means of additional assumptions. For example, Buryachenko and Parton (1992) obtained an expression for $\mathbf{L}^*$ for the ellipsoidal RVE w^{el}. Then the tensor $\mathbf{L}^*$ now has a form (see details of Buryachenko and Parton, 1992)

$$\mathbf{L}^* = \mathbf{L}^{(0)} + \mathbf{R}n\left\{\mathbf{I} - \mathbf{P}(w^{\mathrm{el}})\mathbf{R}n - \sum_{\mathbf{m}\neq\mathbf{0}} \mathbf{T}_{i\mathbf{m}}(\mathbf{x}_i - \mathbf{x}_{\mathbf{m}})\right\}^{-1}, \quad \mathbf{x}_{\mathbf{m}} \in w^{\mathrm{el}}, \tag{5.13}$$

where for the sake of definiteness $\mathbf{x}_i = \mathbf{x}_0 = \mathbf{0}$ and the index $\mathbf{m}$ is a triplet: $\mathbf{m} = (m_1, m_2, m_3)$. It is assumed that $\mathbf{x}_i$ coincides with the center of the region w^{el}, containing a quite large number of inclusions $\mathbf{x}_{\mathbf{m}} \in w^{\mathrm{el}}$. In the particular case of simple cubic packing of the spherical homogeneous inclusions Buryachenko and Parton (1992) showed that the maximum error of the using of the spherical RVE (if the RVE dimensions are more than three times the inclusion spacing) is smaller than 2%. More recently the analogous result was obtained by Rodin (1993) as well as by Molinari and Mouden (1996).

Alternatively in (5.13) one may use a so-called point approximation (see e.g. Kunin, 1983)

$$\mathbf{T}_{ij}(\mathbf{x}_i - \mathbf{x}_j) = \mathbf{U}(\mathbf{x}_i - \mathbf{x}_j)\mathbf{R} \tag{5.14}$$

which is exact for infinitely spaced heterogeneities. Then eqn (5.13) is reduced to

$$\mathbf{L}^* = \mathbf{L}^{(0)} + \mathbf{R}n\left\{\mathbf{I} - \mathbf{P}(w^{\mathrm{el}})\mathbf{R}n - \sum_{\mathbf{m}\neq\mathbf{0}} \mathbf{U}(\mathbf{x}_i - \mathbf{x}_{\mathbf{m}})\mathbf{R}\right\}^{-1}, \quad \mathbf{x}_{\mathbf{m}} \in w^{\mathrm{el}}. \tag{5.15}$$

A significant error of the relation (5.15) as compared to (5.13) was demonstrated by Buryachenko and Parton (1992) (see also Section 6).

6. Numerical results

Let us consider as an example a composite consisting of isotropic homogeneous components and having identical spherical inclusions $\mathbf{L}^{(i)} = (3k^{(i)}, 2\mu^{(i)}) \equiv 3k^{(i)}\mathbf{N}_1 + 2\mu^{(i)}\mathbf{N}_2$, ($\mathbf{N}_1 = \delta \otimes \delta/3$, $\mathbf{N}_2 = \mathbf{I} - \mathbf{N}_1$). For simple cubic (SC) lattice of spherical inclusions the tensor of effective moduli $\mathbf{L}^*$ (5.7) is characterized by three elastic moduli:

$$
\begin{aligned}
k^* &= \tfrac{1}{3}L^*_{1111} + \tfrac{2}{3}L^*_{1122}, \\
\mu^* &= L^*_{1212}, \\
\tilde{\mu}^* &= \tfrac{1}{2}L^*_{1111} - \tfrac{1}{2}L^*_{1122},
\end{aligned}
\tag{6.1}
$$

where the stiffness components are given with respect to a coordinate system whose base vectors are normal to the faces of the unit cell. In the interest of obtaining maximum difference between the effective properties, estimated by the different methods we will consider the examples for hard inclusions ($v^{(0)} = v^{(1)} = 0.3, \mu^{(1)}/\mu^{(0)} = 1000$) as well as for the voids ($\mathbf{L}^{(1)} \equiv \mathbf{0}$), and a number of values of the volume concentration of inclusions. The local elastic moduli (6.1) are computed by analytical method by Sangani and Lu (1987), Nunan and Keller (1984), and Kushch (1987) as well as by the formulae (5.13) and (5.15) (see Tables 1 and 3); here $v^{(i)} \equiv (3k^{(i)} - 2\mu^{(i)})/(6k^{(i)} - 2\mu^{(i)})$, ($i = 0, 1$) is a Poisson ratio.

According to Tables 1 and 3, the error of eqn (5.13) is maximum for $c = 0.5$ and does not exceed 30 and 23% for the rigid inclusions and for the voids, respectively; similar errors for $c = 0.4$ do not exceed 14 and 19%, respectively. Tables 1 and 3 reveal that the method (5.13) performs quite well (at least for $\mathbf{L}^{(1)} \equiv \mathbf{0}$) even if c is quite close to limiting packing coefficient for SC: $c_{\max} = \pi/6 \cong 0.52$. The calculation by the approximate variant (5.15) gives contradictory results for $c > 0.35$: the component L^*_{1111} oscillates around zero as c increases. Tables 1 and 3 give the values $\mathbf{L}^*$ calculated from formulae (5.13) and (5.15) for spherical RVE with radius $r_{25} = 25|\mathbf{e}_1|$, containing 25 layers of inclusions around a considered inclusion v_i. Similar results for elastic moduli $\mathbf{L}^*$ (5.13) estimated for different radii of RVE are represented in Table 2 and 4. The RVE with $r_1 = |\mathbf{e}_1|$, $r_3 = 3|\mathbf{e}_1|$, and $r_6 = 6|\mathbf{e}_1|$ have one, three, and six layers of surrounding inclusions around

Table 1
The overall elastic constants of SC arrays of voids: (SL) Sangani and Lu (1987), (P) point approximation (5.15), (H1) the proposed method (5.13)

	$k^*/k^{(0)}$			$\mu^*/\mu^{(0)}$			$\tilde{\mu}/\mu^{(0)}$		
c	SL	P	H1	SL	P	H1	SL	P	H1
0.10	0.774	0.774	0.774	0.817	0.807	0.812	0.841	0.846	0.841
0.20	0.602	0.604	0.604	0.665	0.610	0.641	0.718	0.743	0.719
0.30	0.464	0.471	0.471	0.554	0.409	0.496	0.608	0.669	0.612
0.40	0.363	0.364	0.364	0.470	0.204	0.379	0.504	0.613	0.512
0.50	0.242	0.276	0.276	0.375	−0.64	0.288	0.394	0.569	0.413

Table 2
The overall elastic constants of SC arrays of voids estimated by the eqn (5.13) for RVE with the radii r_1, r_3, and r_6

	$k^*/k^{(0)}$			$\mu^*/\mu^{(0)}$			$\tilde{\mu}/\mu^{(0)}$		
c	r_1	r_3	r_6	r_1	r_3	r_6	r_1	r_3	r_6
0.10	0.774	0.774	0.774	0.807	0.813	0.813	0.846	0.840	0.841
0.20	0.604	0.604	0.604	0.625	0.646	0.643	0.733	0.715	0.717
0.30	0.471	0.471	0.471	0.465	0.505	0.499	0.637	0.604	0.610
0.40	0.364	0.364	0.364	0.335	0.392	0.383	0.548	0.500	0.508
0.50	0.276	0.276	0.276	0.234	0.305	0.294	0.460	0.395	0.407

Table 3
The overall elastic constants of SC arrays of rigid inclusions: (N) Nunan and Keller (1984) for $c = 0.1$–0.4, (K) Kushch (1987) for $c = 0.5$, (P) point approximation (5.15), (H1) the proposed method (5.13)

	$k^*/k^{(0)}$			$\mu^*/\mu^{(0)}$			$\tilde{\mu}/\mu^{(0)}$		
c	N/K	P	H1	N/K	P	H1	N/K	P	H1
0.10	1.180	1.179	1.179	1.216	1.207	1.213	1.274	1.286	1.269
0.20	1.405	1.403	1.403	1.455	1.410	1.451	1.704	1.897	1.690
0.30	1.706	1.691	1.691	1.766	1.608	1.740	2.35	4.132	2.319
0.40	2.173	2.074	2.074	2.25	1.802	2.120	3.74	−11.8	3.207
0.50	3.503	2.610	2.610	3.14	1.999	2.674	6.49	−2.15	4.334

Table 4
The overall elastic constants of SC arrays of rigid inclusions estimated by the eqn (5.13) for RVE with the radii r_1, r_3, and r_6

	$k^*/k^{(0)}$			$\mu^*/\mu^{(0)}$			$\tilde{\mu}/\mu^{(0)}$		
c	r_1	r_3	r_6	r_1	r_3	r_6	r_1	r_3	r_6
0.10	1.179	1.179	1.179	1.207	1.215	1.214	1.285	1.266	1.268
0.20	1.403	1.403	1.403	1.427	1.458	1.453	1.791	1.667	1.682
0.30	1.691	1.691	1.691	1.682	1.759	1.746	2.708	2.236	2.291
0.40	2.074	2.074	2.074	2.000	2.164	2.134	4.423	2.985	3.131
0.50	2.610	2.610	2.610	2.434	2.774	2.705	7.686	3.858	4.165

the considered one, respectively. As can be seen from the Tables 2 and 4 the estimations of $\mathbf{L}^*$ (5.13) for the RVE with the radii r_3 and r_6 (at $c = 0.5$) differ from presented in the Tables 1 and 3 by 10 and 3% as maximum, respectively, i.e., the RVE with three layers of inclusions can already be considered as representative and the principle of locality by Sokolkin and Tashkinov (1984) holds. The use of six-layers spherical RVE guarantees at least three-digit accuracy for the voids and two-digit accuracy for the rigid inclusions.

The analysis of Tables 1–4 is an extension summary of the work of Buryachenko and Parton (1992), where the stated problem was investigated. More recently an analogous result was obtained by the similar methods by Rodin (1993) as well as by Molinari and Mouden (1996). This example is considered deliberately for the demonstration of high accuracy of the proposed method (5.13) based on the use of the hypothesis (H1). Still, one could argue that if the hypothesis (H1) is used for the estimation of effective nonlocal properties then the accuracy is satisfactory for many purposes, although to our knowledge, the exact analytical methods has never been actually implemented in the estimation of nonlocal effective properties, which we will consider now by the approximate method (4.14) and (5.8).

Now let us compare the different components of the normalized tensor $\mathscr{L}^{*\mathrm{nor}} \equiv 10\mathscr{L}^*/(\mu^{(0)}|\mathbf{e}_1|^2)$ describing the nonlocal properties and obtained by the use of eqns (4.14) and (5.8) for different radii of the RVE: $r_6 = 6|\mathbf{e}_1|$, $r_9 = 9|\mathbf{e}_1|$, $r_{25} = 25|\mathbf{e}_1|$. As can be seen from Table 5 for rigid inclusions the maximum errors are 18% at $r = r_6$ and 7% at $r = r_9$ compared to the result at $r = r_{25}$. Therefore, for good accuracy (i.e. 7% error) of a constitutive nonlocal model, the minimum RVE size is relatively large: $r = 9|\mathbf{e}_1|$. In the case of rigid inclusions the minimum RVE size required for the same accuracy of estimated nonlocal parameters is always substantially larger as compared to the case of voids (see Table 6). Tables 5 and 6 show that for the systems considered, the minimum RVE size increases with increasing inclusion volume fraction. Comparing the analysis of Tables 1–4 to the Tables 5 and 6 shows that for equal accuracy of effective elastic properties for local and nonlocal response, the RVE must be larger in the case of nonlocal properties. This fact is explained by different behavior at infinity of integrand functions $\mathscr{F}(\mathbf{x}_i - \mathbf{y})$ and $\mathscr{F}_2(\mathbf{x}_i - \mathbf{y})$ (4.6) in the integral representatives of the local (4.13), (5.7) and nonlocal (4.14), (5.8) operators, respectively.

As mentioned above the differential representation of the nonlocal operator (5.6) has the

Table 5
The overall normalized nonlocal elastic constants of SC arrays of rigid inclusions estimated by the eqn (5.8) for RVE with the radii r_6, r_9, and r_{25}

	$\mathscr{L}^{*\mathrm{nor}}_{111111}$			$\mathscr{L}^{*\mathrm{nor}}_{112211}$			$\mathscr{L}^{*\mathrm{nor}}_{121211}$		
c	r_6	r_9	r_{25}	r_6	r_9	r_{25}	r_6	r_9	r_{25}
0.10	0.164	0.166	0.167	−0.059	−0.060	−0.060	−0.068	−0.068	−0.069
0.20	0.875	0.894	0.904	−0.303	−0.312	−0.318	−0.233	−0.235	−0.236
0.30	2.588	2.678	2.725	−0.859	−0.898	−0.918	−0.468	−0.471	−0.473
0.40	5.771	6.067	6.226	−1.771	−1.893	−1.958	−0.749	−0.752	−0.754
0.50	10.37	11.10	11.50	−2.700	−2.981	−3.133	−0.997	−1.003	−1.005

Table 6
The overall normalized nonlocal elastic constants of SC arrays of voids estimated by the eqn (5.8) for RVE with the radii r_3, r_6, and r_{25}

	$\mathscr{L}^{*\mathrm{nor}}_{111111}$			$\mathscr{L}^{*\mathrm{nor}}_{112211}$			$\mathscr{L}^{*\mathrm{nor}}_{121211}$		
c	r_3	r_6	r_{25}	r_3	r_6	r_{25}	r_3	r_6	r_{25}
0.10	0.070	0.070	0.070	−0.017	−0.018	−0.018	−0.050	−0.052	−0.053
0.20	0.192	0.192	0.193	−0.041	−0.042	−0.042	−0.137	−0.147	−0.149
0.30	0.320	0.318	0.318	−0.058	−0.058	−0.059	−0.199	−0.211	−0.219
0.40	0.437	0.432	0.431	−0.063	−0.063	−0.064	−0.206	−0.222	−0.232
0.50	0.535	0.527	0.525	−0.054	−0.054	−0.055	−0.156	−0.171	−0.182

disadvantage that is uses the Taylor expansions (4.4) and (4.11) in the RVE about the center $\mathbf{x}$; it is sometimes violated in practice. For instance an infinite or non-existing derivative of some finite order in $\langle\boldsymbol{\varepsilon}\rangle_{\mathbf{x}}(\mathbf{x})$ can take place; the radius of convergence of Taylor series (4.4) and (4.7) can be less than the radius of the RVE, which may cause the slow convergence at an infinite range for integral operators. So in our case, the integral (4.14) is conditionally convergent, i.e. it depends on the shape of the RVE. Similar circumstances can lower the feasibility of the Fourier transform method that is considered in the current paper in sufficient detail deliberately for the demonstration of disadvantages of this popular method. For later use we derived the integral form of the nonlocal operator (4.17) which does not suffer from this limitation, and which will be considered now.

Let for the sake of definiteness the composite materials is subjected to a strain gradient along the direction $\mathbf{e}_3$ and only one component $\varepsilon^{\mathrm{con}}_{ij}$ (5.10) differs from zero:

$$\varepsilon^{\mathrm{con}}_{ij} \neq 0; \quad \text{all other } \varepsilon^{\mathrm{con}}_{kl} \equiv 0 \quad (kl \neq ij). \tag{6.2}$$

Finally, lest it be thought that all nonlocal operators with smooth $\langle\boldsymbol{\varepsilon}\rangle_{\mathbf{x}}(\mathbf{x})$ are straightforward to solve by the method (4.12) and (5.6), we give two counterexamples: a monotonical smooth function

$$f(x_3) = f_1(x_3) \equiv \begin{cases} \dfrac{(x_3/a)^4}{1+(x_3/a)^4} & \text{for } x_3 \geqslant 0, \\ 0 & \text{for } x_3 < 0, \end{cases} \tag{6.3}$$

and an even infinitely differentiable function

$$f(x_3) = f_2(x_3) \equiv 1 - \mathrm{e}^{-(x_3/a)^4}, \tag{6.4}$$

where a is a positive length parameter, and let $c = 0.5$. Clearly $f_3(x_1)$, $f_1''(x_3) \equiv 0$ and $f_2(x_3)$, $f_2''(x_3) = 0$ at $x_3 \leqslant 0$ and $x_3 = 0$, respectively. Therefore, the differential approach (5.6) leads to degenerate results:

$$\langle\boldsymbol{\sigma}\rangle_{\mathbf{x}}(\mathbf{x}) \equiv \mathbf{0} \quad \text{for } x_3 \leqslant 0, \tag{6.5}$$

and

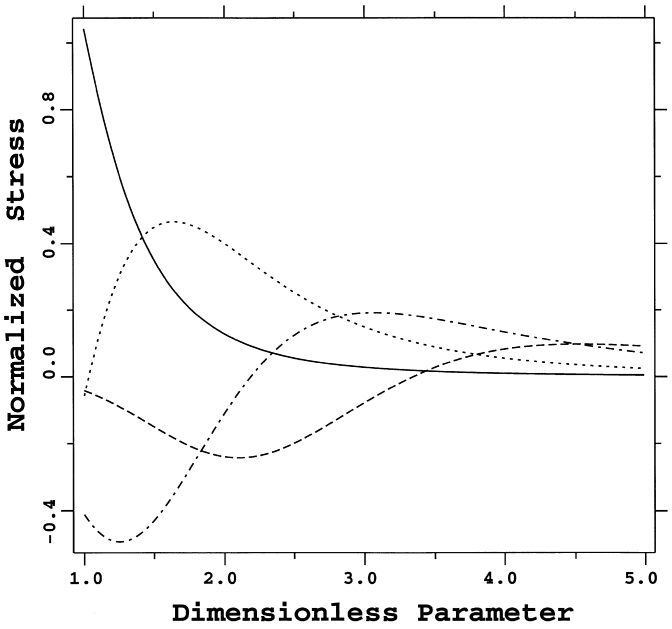

Fig. 1. Normalized nonlocal stresses $\sigma_{33}^{\rm non}(x_3)/(\mu^{(0)}\varepsilon_{33}^{\rm con})$ as functions of the dimensionless parameter $a^{\rm nor}$: $x_3/|\mathbf{e}_3| = 0$ (solid curve), 1 (dotted curve), 2 (dot–dashed curve), 3 (dashed curve).

$$\langle\boldsymbol{\sigma}\rangle_{\mathbf{x}}(\mathbf{x}) = \mathbf{0} \quad \text{for } x_3 = 0, \tag{6.6}$$

for the functions f_1 (6.3) and f_2 (6.4), respectively.

Thus the approximation of a nonlocal operator by the second-order differential operator might be too crude even if the driving function $\{\boldsymbol{\varepsilon}\}_{\mathbf{x}}(\mathbf{x})$ is smooth enough (5.10), (6.3) and (6.4). At the same time, the treatment of indicated functions within the framework of the method of successive approximations (4.17), (5.11) is quite efficient. We illustrate the statement made above with the function $f_1(x_3)$ (6.3) for rigid inclusions (when the estimations obtained by different methods will have the maximum dissimilarity from one another), and a number of values of a. Let the tensors $\boldsymbol{\sigma}^{\rm loc}(\mathbf{x}) \equiv \mathbf{L}^*\langle\boldsymbol{\varepsilon}\rangle_{\mathbf{x}}(\mathbf{x})$ and $\boldsymbol{\sigma}^{\rm non}(\mathbf{x}) \equiv \langle\boldsymbol{\sigma}\rangle_{\mathbf{x}}(\mathbf{x}) - \mathbf{L}^*\langle\boldsymbol{\varepsilon}\rangle_{\mathbf{x}}(\mathbf{x})$ be named the local and nonlocal stresses, respectively. In Fig. 1 the 33-components of normalized nonlocal stresses $\sigma_{33}^{\rm non}(\mathbf{x})/(\mu^{(0)}\varepsilon_{33}^{\rm con})$ are plotted as the functions of dimensionless parameters $a^{\rm nor} \equiv a/|\mathbf{e}_3|$ for different values of normalized coordinates $x^{\rm nor} \equiv x_3/|\mathbf{e}_3| = 0, 1, 2, 3$ (layer number); $c = 0.5$. As can be seen from Fig. 1, the nonlocal stress $\sigma_{33}^{\rm non}(0)/(\mu^{(0)}\varepsilon_{33}^{\rm con}) \neq 0$ and decreases with increasing $a^{\rm nor}$, notwithstanding the fact, that according to the formulae (5.6) and (6.3), $\boldsymbol{\sigma}^{\rm loc}(\mathbf{0}) = \boldsymbol{\sigma}^{\rm non}(\mathbf{0}) \equiv \mathbf{0}$, $\forall a > 0$. At the points $x_3 = |\mathbf{e}_3|$ the nonlocal stress $\sigma_{33}^{\rm non}(\mathbf{x})/(\mu^{(0)}\varepsilon_{33}^{\rm con})$ reaches its maximum at $a^{\rm nor} = 1.65$ and equals 8% of the local stress $\sigma_{33}^{\rm loc}(\mathbf{x})/(\mu^{(0)}\varepsilon_{33}^{\rm con})$ ($x_3 = |\mathbf{e}_3|$). It should be mentioned that $f_1(x_3)$ (6.3) is a monotonic function, however the nonlocal stresses may reverse sign, which is compatible with changing of the sign of the second derivation $f_1''(x_3)$ in the framework of the differential approach (5.6).

For the function $f_2(x_3)$ one obtains a similar dependence of nonlocal stresses on the parameter x_3 and $a^{\rm nor}$. Figure 2 shows the different components of normalized nonlocal stresses $\sigma_{ij}^{\rm non}(\mathbf{x})/(\mu^{(0)}\varepsilon_{ij}^{\rm con})$ ($ij = 11, 33, 13, 12$) as the functions of parameters $a^{\rm nor}$ at $x_3 = 0$, that is in contrast

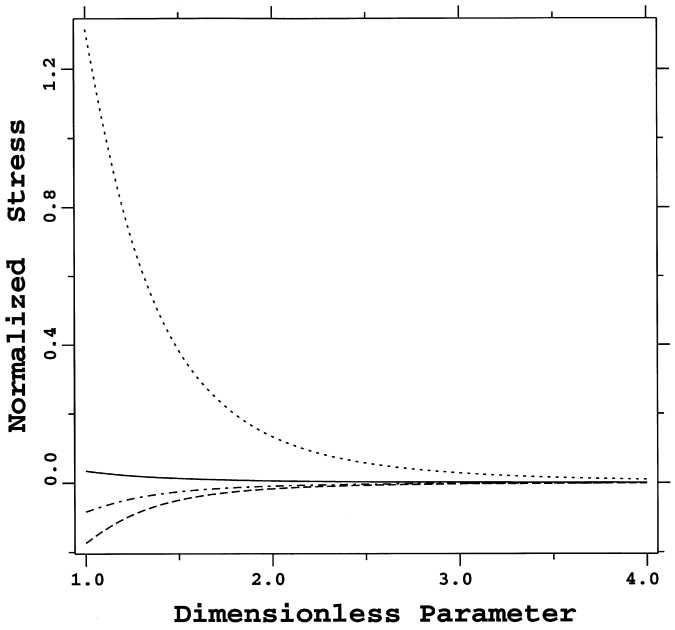

Fig. 2. Normalized nonlocal stresses $\sigma_{ij}^{\rm non}(0)/(\mu^{(0)}\varepsilon_{ij}^{\rm con})$ as functions of the dimensionless parameter $a^{\rm nor}$: $ij = 11$ (solid curve), 33 (dotted curve), 13 (dot–dashed curve), 12 (dashed curve).

to the results obtained by conventional differential approach (5.6): $\boldsymbol{\sigma}^{\rm nor}(\mathbf{0}) \equiv \mathbf{0}$. In Fig. 3 the normalized nonlocal stress $\sigma_{33}^{\rm non}(\mathbf{0})/(\mu^{(0)}\varepsilon_{33}^{\rm con})$ as a function of the parameter $a^{\rm nor}$ are calculated for a number of values of the concentration of inclusions c. Figure 3 shows, that the nonlocal stresses $\sigma_{33}^{\rm non}(\mathbf{0})/(\mu^{(0)}\varepsilon_{33}^{\rm con})$ increase with increasing reinforcement volume fraction c.

Let us now compare the 33-components of nonlocal normalized stresses $\sigma_{33}^{\rm non}(x_3)/(\mu^{(0)}\varepsilon_{33}^{\rm con})$ estimated by both the Fourier transform method (5.6) and by the iteration method (5.9). Figure 4 shows the normalized stresses $\sigma_{33}^{\rm non}(x_3)/(\mu^{(0)}\varepsilon_{33}^{\rm con})$ as the functions of layer numbers $x_n^{\rm nor} \equiv x_{n3}/|\mathbf{e}_3| = 0, \pm 1, \pm 2, \ldots$ $(n = 0, \pm 1, \pm 2, \ldots)$ for the function $f_2(x_3)$ (6.4) with dimensionless parameter values $a^{\rm nor} = 1$ and $a^{\rm nor} = 2$, and $c = 0.5$. It is evident from Fig. 4 that for strongly-varying average strains $\langle \boldsymbol{\varepsilon} \rangle_{\mathbf{x}}(\mathbf{x})$ (for $a^{\rm nor} = 1$) qualitative difference between the results obtained by dissimilar methods occurs. For more smooth fields $\langle \boldsymbol{\varepsilon} \rangle_{\mathbf{x}}(\mathbf{x})$ (for $a^{\rm nor} = 2$) the analogous curves are distinguished from one another, not nearly so much as in a case $a^{\rm nor} = 1$. In so doing at $a^{\rm nor} = 2$ the use of the iteration method leads to the values $\sigma_{33}^{\rm non}(x_{13}) = 1.16\sigma_{33}^{\rm loc}(x_{13})$, $\sigma_{33}^{\rm non}(x_{23}) = 0.15\sigma_{33}^{\rm loc}(x_{23})$, and $\sigma_{33}^{\rm non}(x_{03}) = 0.03\sigma_{33}^{\rm loc}(x_{23})$, although $\sigma_{33}^{\rm loc}(x_{03}) = \sigma_{33}^{\rm non}(x_{03}) \equiv 0, \forall a > 0$ with the use of Fourier transform method.

A slowly-varying function

$$f(x_3) = f_3(x_3) \equiv \cos\left(\frac{\pi x_3}{a}\right) \tag{6.7}$$

used in both methods (5.6) and (5.9) leads to nonlocal normalized stresses $\sigma_{33}^{\rm nor}(x_3)/(\mu^{(0)}\varepsilon_{33}^{\rm con})$ are close to one another for sufficiently large $a^{\rm nor} \geqslant 2$ (see Fig. 5).

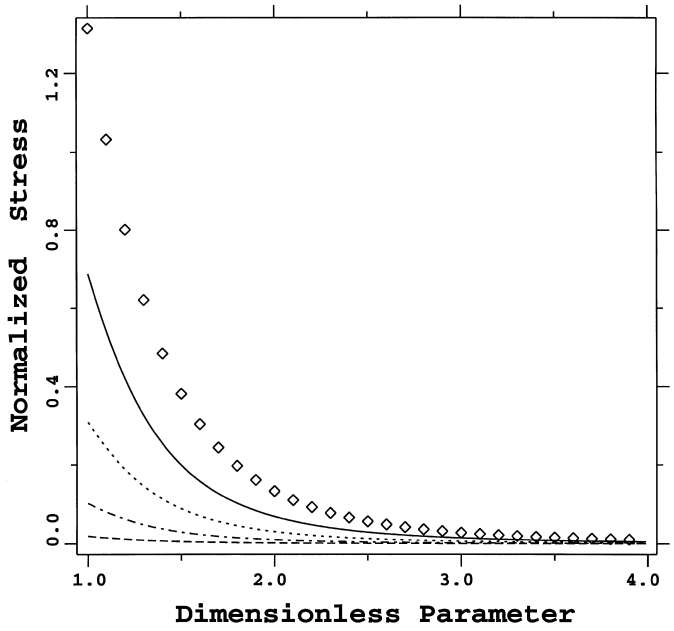

Fig. 3. The 11-component of normalized nonlocal stresses $\sigma_{33}^{\mathrm{non}}(0)/(\mu^{(0)}\varepsilon_{33}^{\mathrm{con}})$ as functions of the dimensionless parameter a^{nor} at the different concentration of rigid inclusions: $c = 0.5$ ($\diamond$), $c = 0.4$ (solid curve), 0.3 (dotted curve), 0.2 (dot–dashed curve), 0.1 (dashed curve).

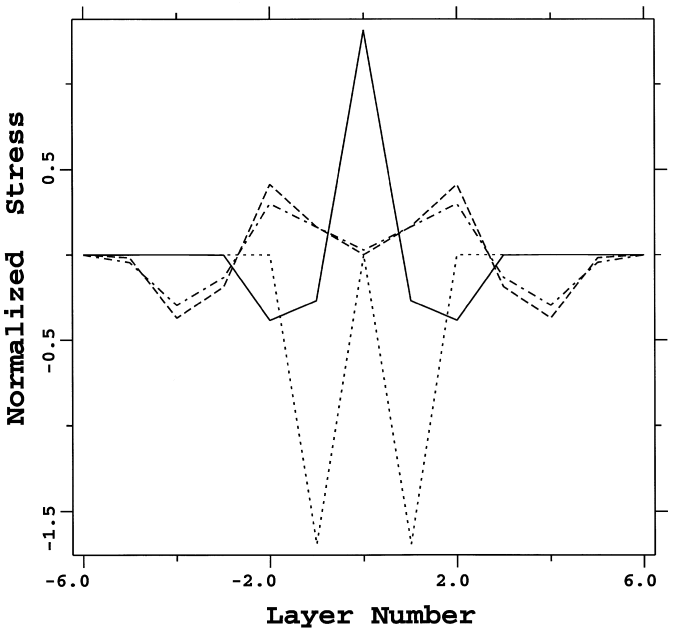

Fig. 4. Normalized nonlocal stresses $\sigma_{33}^{\mathrm{non}}(\mathbf{x})/(\mu^{(0)}\varepsilon_{33}^{\mathrm{con}})$ as functions of the normalized coordinates x^{nor} calculated by the use of the Fourier method (5.6) (dotted and dashed curves) as well as by the iteration method (5.9) (solid and dot–dashed curves) for $a^{\mathrm{nor}} = 1$ (solid and dotted curves) and $a^{\mathrm{nor}} = 2$ (dashed and dot–dashed curves).

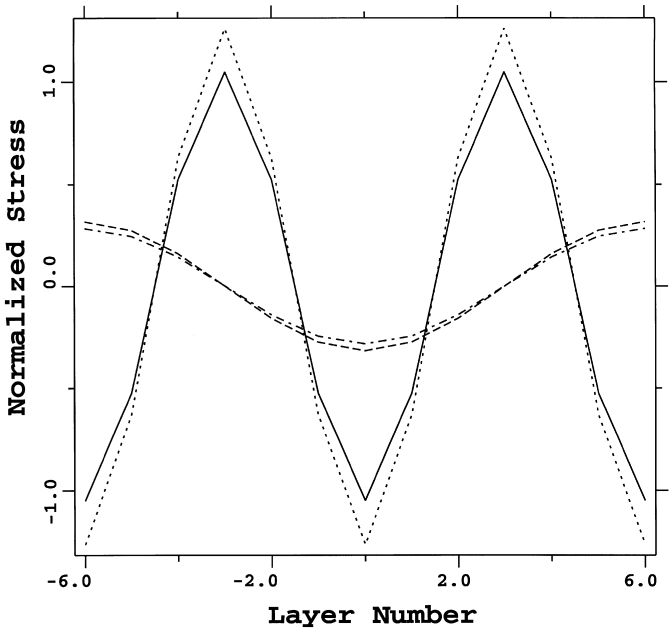

Fig. 5. Normalized nonlocal stresses $\sigma_{33}^{\rm non}(\mathbf{x})/(\mu^{(0)}\varepsilon_{33}^{\rm con})$ as functions of the normalized coordinates $x^{\rm nor}$ calculated by the use of the Fourier method (5.6) (dotted and dashed curves) as well as by the iteration method (5.9) (solid and dot–dashed curves) for $a^{\rm nor}=1$ (solid and dotted curves) and $a^{\rm nor}=2$ (dashed and dot–dashed curves).

In conclusion it may be said that the relations obtained depend on the values associated with the mean distance between inclusions and do not depend on the other characteristic size, i.e. the mean inclusion diameter. This fact may be explained by the initial acceptance of the hypothesis (H1) dealing with the homogeneity of the field $\bar{\boldsymbol{\sigma}}(\mathbf{x})$ inside each inclusion. In the case of a variable representation of $\bar{\boldsymbol{\sigma}}(\mathbf{x})$ $(\mathbf{x}\in v_i)$, for instance in polynomial form, the mean size of the inclusions will be contained in the nonlocal dependence of microstresses on the average stress $\langle\boldsymbol{\sigma}\rangle_{\mathbf{x}}(\mathbf{x})$.

It should be mentioned that the effective constitutive eqn (5.5) was derived for points $\mathbf{x}_i$ located sufficiently far from the boundary of the body ∂w. In so doing the relations developed have been obtained by the use of the whole-space Green's function (3.3). Then use of nonlocal constitutive relations (5.5) requires more complicated boundary conditions (see for details Kunin and Vaisman, 1970; Beran and McCoy, 1970; Drugan and Willis, 1996); this question is beyond the scope of the current study.

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